INVARINANCE UNDER CORDIALITY OF PATH UNION OF $C_5\Theta P_k$

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Abstract:
We obtain different structures of $P_m(G)$ for $G = C_5\Theta P_k$. We take $k=2,3,4,5,6$. The different structures are obtained on path union because we use different vertices on $C_5\Theta P_k$ to construct $P_m(G)$. We show all structures so obtained are cordial. Hence invariance under cordiality.

Keywords: Invariance; Cordial Labeling; Path Union; Path; Cycle.


1. Introduction

The graphs we consider are simple, finite, undirected and connected. For terminology and definitions we depend on Graph Theory by Harary [6], A dynamic survey of graph labeling by J.Gallian [8] and Douglas West [9]. I.Cahit introduced the concept of cordial labeling [5]. $F: V(G) \rightarrow \{0, 1\}$ is a function. From this label of any edge $(uv)$ is given by $|f(u) - f(v)|$. Further number of vertices labeled with 0 i.e. $v(0)$ and the number of vertices labeled with 1 i.e. $v(1)$ differ at most by one. Similarly number of edges labeled with 0 i.e. $e(0)$ and number of edges labeled with 1 i.e. $e(1)$ differ at most by one. Then the function $f$ is called as cordial labeling. I.Cahit has shown that: every tree is cordial; $K_n$ is cordial if and only if $n \leq 3$; $K_{m,n}$ is cordial for all $m$ and $n$; the friendship graph $C_3(t)$ (i.e., the one-point union of $t$ copies of $C_3$) is cordial if and only if $t$ is not congruent to $2$ (mod 4); all fans are cordial; the wheel $W_n$ is cordial if and only if $n$ is not congruent to $3$ (mod 4). A lot of work has been done in this type of labeling. One may refer dynamic survey by J. Gallian [8].

Our focus of attention is on path unions on different graphs. For a given graph there are different path unions (up to isomorphism) structures possible. It depends on which point on $G$ is used to fuse with vertex of $P_m$ to obtain path-union. We have shown that for $G = bull$ on $C_3$, bull on $C_4$, $C_3^+$, $C_4^-$ then different path union $P_m(G)$ are cordial [4]. It is called as invariance under cordial labeling. We use the convention that $v(0,1) = (a,b)$ to indicate the number of vertices labeled with 0 are $a$ in number and that number of vertices labeled with 1 are $b$ in number. Further $e(0,1) = (x,y)$ we mean the number of edges labeled with 0 are $x$ and number of edges labeled with 1 are $y$ in number. The graph whose cordial labeling is known is cordial graph.
2. Preliminaries

**Fusion of vertex** Let G be a (p,q) graph. Let u≠v be two vertices of G. We replace them with single vertex w and all edges incident with u and that with v are made incident with w. If a loop is formed is deleted. The new graph has p-1 vertices and at least q-1 edges. [9] Path union of G , i.e. P_m(G) is obtained by taking a path p_m and take m copies of graph G Then fuse a copy each of G at every vertex of path at given fixed point on G. It has mp vertices and mq +m-1 edges. Where G is a (p,q) graph.

3. Theorems Proved

3.1. Theorem: All structures of P_m(G) are cordial for G = C_5\oplus P_2

Proof: Define f: V (P_m(G))→{0,1} as follows. f gives two labeled copies of C_5\oplus P_2 Type A and Type B. These are shown in fig 5.1 and fig 5.2 below.

![Fig 5.1](image1.png)

![Fig 5.2](image2.png)

There are two structures possible on P_m(G). In **structure 1** we choose vertex ‘a’ from type A and ‘b’ from Type B to fuse on vertex of path P_m =(v_1, v_2, v_3, …v_m). In **structure 2** we choose vertex ‘c’ from type A and vertex ‘d’ from Type B to fuse on vertex of path P_m. Note that any of the 5 pendent vertices each from Type A and that from Type B will be vertex ‘a’ or vertex ‘b’ respectively. And any of the 5 degree three vertices from each type A and Type B will be vertex ‘c’ and ‘d’ respectively.

![Fig 5.3](image3.png)

**Structure 1**: We start with a unlabeled path P_m and at vertex v_i on it fuse vertex ‘a’ from type A if i≡1, 2(mod 4) and vertex ‘b’ from type b if i≡2, 3 (mod 4). The label of edge (aa) = label of edge (bb)= 0 and label of edge (ab) = 1. The label numbers in this case are v_i(0,1)=(5m,5m) for all m
and for edges when \( m = 2x, x = 1, 2, \ldots \) we have \( e_i(0,1) = (5m+x,5m+x-1) \) and when \( m \) is of type \( 2x+1, x = 0, 1, 2 \ldots \) we have \( e_i(0,1) = (5m+x,5m+x) \).

For **structure 2** we repeat the same procedure as above. We start with an unlabeled path \( P_m \) and at vertex \( v_i \) on it fuse vertex ‘c’ from type A if \( i = 1, 2 \) (mod 4) and vertex ‘d’ from type B if \( i = 2, 3 \) (mod 4). The label of edge (cc) = label of edge (dd) = 0 and label of edge (cd) = 1. The label numbers in this case are \( v_i(0,1) = (5m,5m) \) for all \( m \) and for edges when \( m = 2x, x = 1, 2, \ldots \) we have \( e_i(0,1) = (5m+x,5m+x-1) \) and when \( m \) is of type \( 2x+1, x = 0, 1, 2 \ldots \) we have \( e_i(0,1) = (5m+x,5m+x) \).

The graph is cordial.

### 3.2. Theorem: All structures of \( P_m(G) \) are cordial for \( G = C_5 \Theta P_3 \).

Proof: Define \( f: V(P_m(G)) \rightarrow \{0,1\} \) as follows. \( f \) gives two labeled copies of \( C_5 \Theta P_2 \) Type A and Type B. These are shown in Fig 5.3 and Fig 5.4 below.

![Fig 5.4](image1.png) \( v_i(0,1) = (8,7), e_i(0,1) = (8,7) \)

![Fig 5.5](image2.png) \( v_i(0,1) = (7,8), e_i(0,1) = (7,8) \)

Different structures on path union are due to we use different vertices on \( G \) namely ‘a’, ‘b’ and ‘c’ to fuse with vertex of \( P_m = (v_1, v_2, \ldots v_m) \)

In **structure 1** we fuse vertex ‘a’ from Type A with vertex \( v_i \) of \( P_m \) and vertex ‘d’ from Type B with vertex \( v_i \) of \( P_m \) alternately starting with type A. On \( P_m \) the label of edge (ad) = 1 and label(da) = 1. The label number distribution on resultant graph are \( v_i(0,1) = (8+15x,7+15x) \) when \( m \) is odd number given by \( 2x+1, x = 0, 1, 2 \ldots \), and \( v_i(0,1) = (15x,15x) \) when \( m \) is even number given by \( 2x, x = 1, 2 \ldots \). For all \( m \) \( e_i(0,1) = (8m, 8m-1) \).

In **structure 2** we fuse vertex ‘b’ from Type A with vertex \( v_i \) of \( P_m \) and vertex ‘e’ from Type B with vertex \( v_i \) of \( P_m \) alternately starting with type A. On \( P_m \) the label of edge (be) = 1 and label(eb) = 1. The label number distribution on resultant graph are \( v_i(0,1) = (8+15x,7+15x) \) when \( m \) is odd number given by \( 2x+1, x = 0, 1, 2 \ldots \), and \( v_i(0,1) = (15x,15x) \) when \( m \) is even number given by \( 2x, x = 1, 2 \ldots \). For all \( m \) \( e_i(0,1) = (8m, 8m-1) \).

In **structure 3** we fuse vertex ‘c’ from Type A with vertex \( v_i \) of \( P_m \) and vertex ‘f’ from Type B with vertex \( v_i \) of \( P_m \) alternately starting with type A. On \( P_m \) the label of edge (be) = 1 and label(ef) = 1. The label number distribution on resultant graph are \( v_i(0,1) = (8+15x,7+15x) \) when \( m \) is odd number given by \( 2x+1, x = 0, 1, 2 \ldots \), and \( v_i(0,1) = (15x,15x) \) when \( m \) is even number given by \( 2x, x = 1, 2 \ldots \). For all \( m \) \( e_i(0,1) = (8m, 8m-1) \). Thus the graph is cordial on all of it’s structures.
3.3. Theorem: All structures of $P_m(G)$ are cordial for $G = C_5 \otimes P_4$

Proof: Define $f: V(P_m(G)) \rightarrow \{0,1\}$ as follows. $f$ gives two labeled copies of $C_5 \otimes P_4$ Type A and Type B. These are shown in fig 5.6 and fig 5.7 below. They have equal label numbers but different distribution.

[Diagram of two labeled graphs]

In **structure 1** we fuse vertex ‘a’ from Type A with vertex $v_i$ of $P_m$ if $i \equiv 1, 2 (mod 4)$ and vertex ‘x’ from Type B with vertex $v_i$ of $P_m$ if $i \equiv 2,3 (mod 4)$. On $P_m$ the label of edge (ax) =1 and label(xa) =1,label of edge (xx) =0 and label(aa) =0. The label number distribution on resultant graph are $v_i(0,1)=(10m,10m)$ for all m and when m is odd number given by $2x+1,x= 0, 1, 2..$ , And $v_i(0,1)=(10m+x,10m+x)$ when m is even number given by $2x,x= 1, 2, ..$ for all m $e_i(0,1) = (10m, 10m-1)$.

In **structure 2** we fuse vertex ‘b’ from Type A with vertex $v_i$ of $P_m$ if $i \equiv 1, 2 (mod 4)$ and vertex ‘y’ from Type B with vertex $v_i$ of $P_m$ if $i \equiv 2,3 (mod 4)$. On $P_m$ the label of edge (by) =1 and label(yb) =1,label of edge (yy) =0 and label(bb) =0. The label number distribution on resultant graph are $v_i(0,1)=(10m,10m)$ for all m and when m is odd number given by $2x+1,x= 0, 1, 2..$ , And $v_i(0,1)=(10m+x,10m+x)$ when m is even number given by $2x,x= 1, 2, ..$ for all m $e_i(0,1) = (10m, 10m-1)$.

In **structure 3** we fuse vertex ‘c’ from Type A with vertex $v_i$ of $P_m$ if $i \equiv 1,2 (mod 4)$ and vertex ‘z’ from Type B with vertex $v_i$ of $P_m$ if $i \equiv 2,3 (mod 4)$. On $P_m$ the label of edge (cz) =1 and label(xc) =1,label of edge (zz) =0 and label(cc) =0. The label number distribution on resultant graph are $v_i(0,1)=(10m,10m)$ for all m and when m is odd number given by $2x+1,x= 0, 1, 2..$ , And $v_i(0,1)=(10m+x,10m+x)$ when m is even number given by $2x,x= 1, 2, ..$ for all m $e_i(0,1) = (10m, 10m-1)$.

In **structure 4** we fuse vertex ‘q’ from Type B with vertex $v_i$ of $P_m$ if $i \equiv 1, 2 (mod 4)$ and vertex ‘d’ from Type A with vertex $v_i$ of $P_m$ if $i \equiv 2,3 (mod 4)$. On $P_m$ the label of edge (qd) =1 and label(dq) =1,label of edge (qq) =0 and label(dd) =0. The label number distribution on resultant graph are $v_i(0,1)=(10m,10m)$ for all m and when m is odd number given by $2x+1,x= 0, 1, 2..$ . And $v_i(0,1)=(10m+x,10m+x)$ when m is even number given by $2x,x= 1, 2, ..$ for all m $e_i(0,1) = (10m, 10m-1)$. The resultant graph structures of $P_m(G)$ are cordial.
3.4. Theorem: All Structures of $P_m(G)$ Are Cordial for $G = C_5 \circ P_2$.

Proof: Define $f$: $V(P_m(G)) \rightarrow \{0,1\}$ as follows. $f$ gives two labeled copies of $C_5 \circ P_2$ Type A and Type B. These are shown in fig 5.8 and fig 5.9 below.

Different structures on path union are due to we use different vertices on $G$ namely ‘e’, ‘a’, ‘b’, ‘c’, ‘d’ to fuse with vertex of $P_m = (v_1, v_2, \ldots, v_m)$.

In **structure 1** we fuse vertex ‘e’ from Type A with vertex $v_i$ of $P_m$ and vertex ‘x’ from Type B with vertex $v_i$ of $P_m$ alternately starting with type A. On $P_m$ the label of edge (ex) =1 and label(xx) =0 and label(ex)=0. The label number distribution on resultant graph are $v_i(0,1)=(13+25x,12+25x)$ when $m$ is odd number given by $2x+1, x= 0, 1, 2, \ldots$, And $v_i(0,1)=(25x,25x)$ when $m$ is even number given by $2x$, $x= 1, 2, \ldots$. For all $m e_i(0,1) = (13m, 13m-1)$.

In **structure 2** we fuse vertex ‘a’ from Type A with vertex $v_i$ of $P_m$ and vertex ‘y’ from Type B with vertex $v_i$ of $P_m$ alternately starting with type A. On $P_m$ the label of edge (ay) =1 and label(ay) =0 and label(yy) =0. The label number distribution on resultant graph are $v_i(0,1)=(13+25x,12+25x)$ when $m$ is odd number given by $2x+1, x= 0, 1, 2, \ldots$, And $v_i(0,1)=(25x,25x)$ when $m$ is even number given by $2x$, $x= 1, 2, \ldots$. For all $m e_i(0,1) = (13m, 13m-1)$.

In **structure 3** we fuse vertex ‘b’ from Type A with vertex $v_i$ of $P_m$ and vertex ‘z’ from Type B with vertex $v_i$ of $P_m$ alternately starting with type A. On $P_m$ the label of edge (bz) =1 and label(bz) =0 and label(bz) =0. The label number distribution on resultant graph are $v_i(0,1)=(13+25x,12+25x)$ when $m$ is odd number given by $2x+1, x= 0, 1, 2, \ldots$, And $v_i(0,1)=(25x,25x)$ when $m$ is even number given by $2x$, $x= 1, 2, \ldots$. For all $m e_i(0,1) = (13m, 13m-1)$.

In **structure 4** we fuse vertex ‘c’ from Type A with vertex $v_i$ of $P_m$ and vertex ‘u’ from Type B with vertex $v_i$ of $P_m$ alternately starting with type A. On $P_m$ the label of edge (uc) =1 and label(uc) =0 and label(uc) =0. The label number distribution on resultant graph are $v_i(0,1)=(13+25x,12+25x)$ when $m$ is odd number given by $2x+1, x= 0, 1, 2, \ldots$, And $v_i(0,1)=(25x,25x)$ when $m$ is even number given by $2x$, $x= 1, 2, \ldots$. For all $m e_i(0,1) = (13m, 13m-1)$.
In structure 5 we fuse vertex ‘d’ from Type A with vertex $v_i$ of $P_m$ and vertex ‘v’ from Type B with vertex $v_i$ of $P_m$ alternately starting with type A. On $P_m$ the label of edge $(dv) = 1$ and label$(vd) = 1$, label of edge $(dd) = 0$ and label$(vv) = 0$. The label number distribution on resultant graph are $v_i(0,1) = (13+25x,12+25x)$ when $m$ is odd number given by $2x+1, x= 0, 1, 2..$. And $v_i(0,1) = (25x,25x)$ when $m$ is even number given by $2x, x= 1, 2, ..$. For all $m e_i(0,1) = (13m, 13m-1)$.

3.5. Theorem: All structures of $P_m(G)$ are cordial for $G = C_5 \Theta P_6$.

Proof: Define $f$: $V(P_m(G)) \to \{0,1\}$ as follows. $f$ gives a labeled copy of $C_5 \Theta P_2$ namely Type A as shown below. We use it repeatedly in a specific way as shown below to obtain labeled copy of $G$.

![Diagram](http://www.ijetmr.com)

We get 6 different structures on path union depending on the point on $C_5 \square P_6$ fused with vertex of path $P_m$ to obtain $P_m(G)$. We start with an unlabeled copy of $P_m$.

**Structure1:** At vertex $v_i$ of $P_m$ fuse a labeled copy as above (type A) at point ‘a’ on it for $i \equiv 1, 2$ (mod 4) and at vertex ‘x’ if $i \equiv 0, 3$ (mod 4). The label number distribution is $v_i(0,1) = (15m, 15m)$ for all $m$. And on edges when $m = 2x+1, x= 0, 1, .. e_i(0,1) = (15+x, 15+x)$ and if $m = 2x, x= 1, 2, .. e_i(0,1) = (15+x, 15+x-1)$.

**Structure2:** At vertex $v_i$ of $P_m$ fuse a labeled copy as above (type A) at point ‘b’ on it for $i \equiv 1, 2$ (mod 4) and at vertex ‘y’ if $i \equiv 0, 3$ (mod 4). The label number distribution is $v_i(0,1) = (15m, 15m)$ for all $m$. And on edges when $m = 2x+1, x= 0, 1, .. e_i(0,1) = (15+x, 15+x)$ and if $m = 2x, x= 1, 2, .. e_i(0,1) = (15+x, 15+x-1)$.

**Structure3:** At vertex $v_i$ of $P_m$ fuse a labeled copy as above (type A) at point ‘c’ on it for $i \equiv 1, 2$ (mod 4) and at vertex ‘z’ if $i \equiv 0, 3$ (mod 4). The label number distribution is $v_i(0,1) = (15m, 15m)$ for all $m$. And on edges when $m = 2x+1, x= 0, 1, .. e_i(0,1) = (15+x, 15+x)$ and if $m = 2x, x= 1, 2, .. e_i(0,1) = (15+x, 15+x-1)$.

**Structure4:** At vertex $v_i$ of $P_m$ fuse a labeled copy as above (type A) at point ‘d’ on it for $i \equiv 1, 2$ (mod 4) and at vertex ‘u’ if $i \equiv 0, 3$ (mod 4). The label number distribution is $v_i(0,1) = (15m, 15m)$ for all $m$. And on edges when $m = 2x+1, x= 0, 1, .. e_i(0,1) = (15+x, 15+x)$ and if $m = 2x, x= 1, 2, .. e_i(0,1) = (15+x, 15+x-1)$.

**Structure5:** At vertex $v_i$ of $P_m$ fuse a labeled copy as above (type A) at point ‘e’ on it for $i \equiv 1, 2$ (mod 4) and at vertex ‘v’ if $i \equiv 0, 3$ (mod 4). The label number distribution is $v_i(0,1) = (15m, 15m)$.
for all m, and on edges when m = 2x + 1, x = 0, 1, \ldots, \epsilon f(0, 1) = (15 + x, 15 + x) and if m = 2x, x = 1, 2, \ldots, \epsilon f(0, 1) = (15 + x, 15 + x - 1).

**Structure 6:** At vertex v_i of P_m fuse a labeled copy as above (type A) at point ‘f’ on it for i \equiv 1, 2 (mod 4) and at vertex ‘w’ if i \equiv 0, 3 (mod 4). The label number distribution is v_f(0, 1) = (15m, 15m) for all m, and on edges when m = 2x + 1, x = 0, 1, \ldots, \epsilon f(0, 1) = (15 + x, 15 + x) and if m = 2x, x = 1, 2, \ldots, \epsilon f(0, 1) = (15 + x, 15 + x - 1). Thus the graph is cordial for all non-isomorphic structures on path union.

4. **Conclusions**

To obtain a path union we generally fuse a copy of G at a fixed point on G with every vertex of P_m. We have considered every vertex of C_5 \ast P_k that will produce non-isomorphic structure on path union. We take k = 2, 3, 4, 5 and 6. We take all path unions and show that all are cordial. This is also called as invariance under cordiality of path union.

**References**

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